CSCI567 Machine Learning (Fall 2021)

Prof. Haipeng Luo

U of Southern California

Nov 18, 2021

Administration

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- duration is 2.5 hours + 10 extra minutes for uploading; *$x\%$ penalty for x minutes late (past 7:40)*.

More on Quiz 2

Coverage: SVM + topics after Quiz 1; some other basic concepts (e.g. training error, regularization, kernel, etc.) might appear in conjunction.

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Same tip: expect variants of questions from discussion/homework

Outline

1. Review of last lecture
2. Multi-armed Bandits
3. Reinforcement learning

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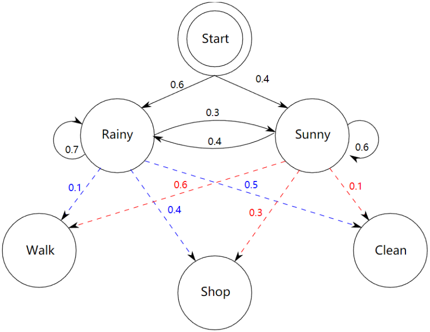
Hidden Markov Models

Model parameters:

- **initial distribution**
  $P(Z_1 = s) = \pi_s$
- **transition distribution**
  $P(Z_{t+1} = s' \mid Z_t = s) = a_{s,s'}$
- **emission distribution**
  $P(X_t = o \mid Z_t = s) = b_{s,o}$



Baum–Welch algorithm

Step 0 Initialize the parameters $(\boldsymbol{\pi}, \boldsymbol{A}, \boldsymbol{B})$

Step 1 (E-Step) Fixing the parameters, compute forward and backward messages for all sample sequences, then use these to compute $\gamma_s^{(n)}(t)$ and $\xi_{s,s'}^{(n)}(t)$ for each n, t, s, s'.

Step 2 (M-Step) Update parameters:

$$\pi_s \propto \sum_n \gamma_s^{(n)}(1), \quad a_{s,s'} \propto \sum_n \sum_{t=1}^{T-1} \xi_{s,s'}^{(n)}(t), \quad b_{s,o} \propto \sum_n \sum_{t:x_t=o} \gamma_s^{(n)}(t)$$

Step 3 Return to Step 1 if not converged

Viterbi Algorithm

Viterbi Algorithm

For each $s \in [S]$, compute $\delta_s(1) = \pi_s b_{s,x_1}$.

For each $t = 2, \ldots, T$,

- for each $s \in [S]$, compute

$$\delta_s(t) = b_{s,x_t} \max_{s'} a_{s',s} \delta_{s'}(t-1)$$

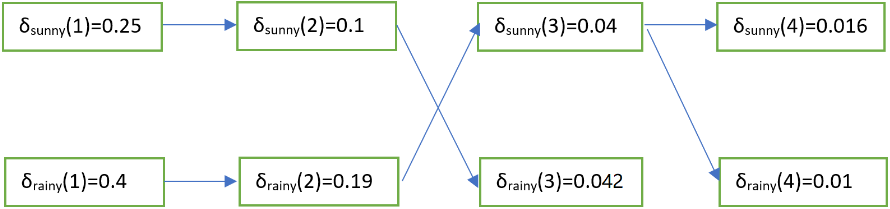
$$\Delta_s(t) = \operatorname*{argmax}_{s'} a_{s',s} \delta_{s'}(t-1)$$

Backtracking: let $z_T^* = \operatorname{argmax}_s \delta_s(T)$.
For each $t = T, \ldots, 2$: set $z_{t-1}^* = \Delta_{z_t^*}(t)$.

Output the most likely path $z_1^*, \ldots, z_T^*$.

Example

Arrows represent the "argmax", i.e. $\Delta_s(t)$.



The most likely path is **"rainy, rainy, sunny, sunny".**

Viterbi Algorithm with missing data

Viterbi Algorithm with partial data $x_{1:T_0}$

For each $s \in [S]$, compute $\delta_s(1) = \pi_s b_{s,x_1}$.

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- for each $s \in [S]$, compute

$$\delta_s(t) = \begin{cases} b_{s,x_t} \max_{s'} a_{s',s}\delta_{s'}(t-1) & \text{if } t \le T_0 \\ \max_{s'} a_{s',s}\delta_{s'}(t-1) & \text{else} \end{cases}$$

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1. Review of last lecture
2. Multi-armed Bandits
   - Online decision making
   - Motivation and setup
   - Exploration vs. Exploitation
3. Reinforcement learning

Decision making

Problems we have discussed so far:

- start with a training dataset
- learn a predictor or discover some patterns

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- receive some feedback
- repeat

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Broadly, these are called **online decision making problems**.

Examples

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- make another move...

Two formal setups

We discuss two such problems today:

- **multi-armed bandit**
- **reinforcement learning**

Mulit-armed bandits: motivation

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Of course there are many slot machines in the casino

- like a bandit with multiple arms (hence the name)
- **if I can play for 10 times, which machines should I play?**

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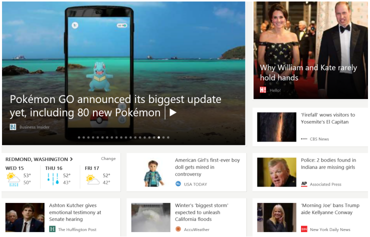

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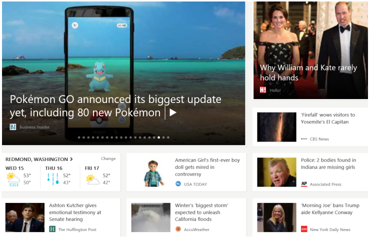

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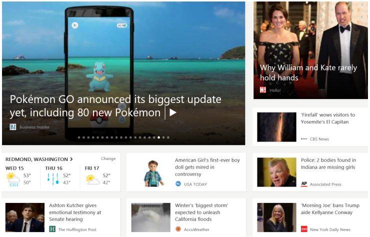
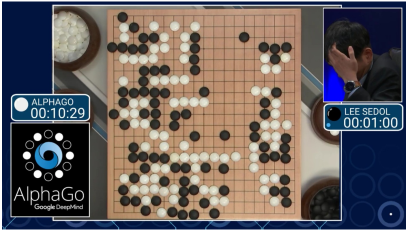
- recommendation systems, each product/movie/news story is an arm (Microsoft MSN indeed employs a variant of bandit algorithm)
- game playing, each possible move is an arm (AlphaGo indeed has a bandit algorithm as one of the components)





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This kind of limited feedback is now usually referred to as bandit feedback

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This is called the **regret**: *how much I regret for not sticking with the best fixed arm in hindsight?*

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- each arm has a different mean $(\mu_1, \ldots, \mu_K)$; the problem is essentially about finding the best arm $\text{argmax}_a \mu_a$ as quickly as possible

Empirical means

Let $\hat{\mu}_{t,a}$ be the **empirical mean** of arm a up to time t:

$$\hat{\mu}_{t,a} = \frac{1}{n_{t,a}} \sum_{\tau \le t : a_\tau = a} r_{\tau,a}$$

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Concentration: $\hat{\mu}_{t,a}$ should be close to μ_a if $n_{t,a}$ is large

Exploitation only

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- the algorithm will never pick arm 1 again!

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We next discuss **three ways** to trade off exploration and exploitation for our simple multi-armed bandit setting.

A natural first attempt

Explore–then–Exploit

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Parameter T_0 clearly controls the exploration/exploitation trade-off

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- clearly it won't work if the environment is changing

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ϵ-Greedy

Pick each arm once for the first K rounds.

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Is there a *more adaptive* way to explore?

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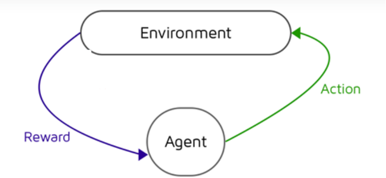
This principle is useful for many other bandit problems.

Outline

Motivation

Multi-armed bandit is among the simplest decision making problems with limited feedback.



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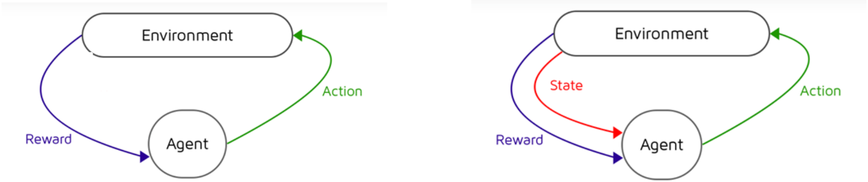

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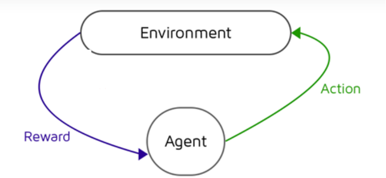

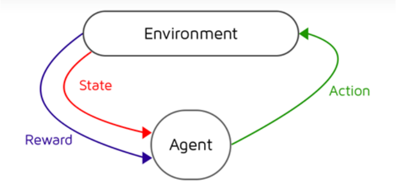

It's often too simple to capture many real-life problems. One thing it fails to capture is the "state" of the learning agent, which has impacts on the reward of each action.

- e.g. for Atari games, after making one move, the agent moves to a different state, with possible different rewards for each action

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The foundation of RL is **Markov Decision Process (MDP)**,
a combination of Markov model (Lec 10) and multi-armed bandit

Markov decision process

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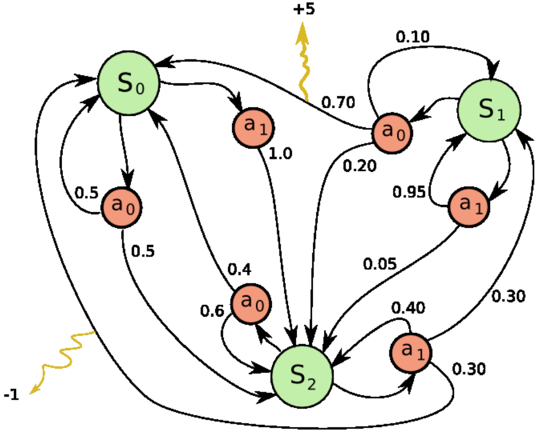
Different from Multi-armed bandit, the reward depends on the state.

Example

3 states, 2 actions



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Note: the discount factor allows us to consider an infinite learning process

Optimal policy and Bellman equation

First goal: knowing all parameters, *how to find the optimal policy*

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V is called the **(optimal) value function**. It satisfies the above **Bellman equation**: $|\mathcal{S}|$ nonlinear equations with $|\mathcal{S}|$ unknowns, *how to solve it?*

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Knowing V, the optimal policy π^* is simply

$$\pi^*(s) = \operatorname*{argmax}_{a \in \mathcal{A}} \left(r_s(a) + \gamma \sum_{s' \in \mathcal{S}} P_a(s, s') V(s') \right)$$

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So the distance between V_k and V is shrinking *exponentially fast*.

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We discuss examples from two families of learning algorithms:

- **model-based** approaches
- **model-free** approaches

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Having estimates of the parameters we can then apply value iteration to find the optimal policy.

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Model-free approaches learn the Q function directly from samples.

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α is like the learning rate

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- update the Q function

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for some learning rate α.

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There are many different algorithms and RL is an active research area.

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A brief introduction to some online decision making problems:

- **Multi-armed bandits**

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  - learning the optimal policy with a known MDP: **value iteration**
  - learning the optimal policy with an unknown MDP: model-based approach and model-free approach (e.g. **Q-learning**)