CSCI567 Machine Learning (Fall 2025)
Viterbi Algorithm Exercises

Haipeng Luo

University of Southern California

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Exercise 1

What is the most likely sequence $z^*_{1:T_0}$ given $x_{1:T_0}$ for some $T_0 < T$?

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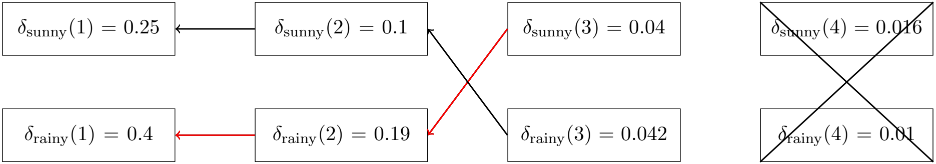

The answer for $T_0 = 3$ is: rainy, rainy, sunny?

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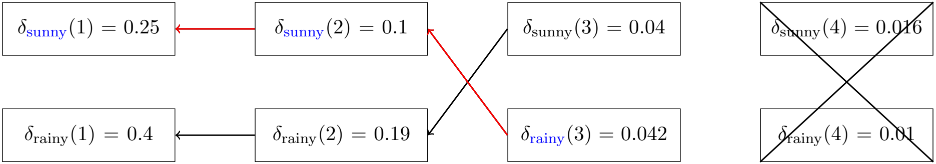
- Is it the first T_0 outputs of the Viterbi algorithm (with all data)?

No. It should be

- $z^*_{T_0} = \operatorname{argmax}_s \delta_s(T_0)$
- for each $t = T_0, \ldots, 2$: $z^*_{t-1} = \Delta_{z^*_t}(t)$



The answer for $T_0 = 3$ is: **"sunny, sunny, rainy"**

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- Is it the same as Exercise 1?

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(imagine you hear a sentence but only care about the first few words)

- Is it the same as Exercise 1?
- Is it the first T_0 outputs of the Viterbi algorithm (with all data)?

Neither. It should be

- $z^*_{T_0} = \mathrm{argmax}_s \, \delta_s(T_0)\beta_s(T_0)$
- for each $t = T_0, \ldots, 2$: $z^*_{t-1} = \Delta_{z^*_t}(t)$

Exercise 2 (cont.)

Reasoning:

$$z^*_{T_0} = \operatorname*{argmax}_{s} \max_{z_{1:T_0-1}} P(Z_{T_0} = s, Z_{1:T_0-1} = z_{1:T_0-1}, X_{1:T} = x_{1:T})$$

Exercise 2 (cont.)

Reasoning:

$$
\begin{aligned}
z^*_{T_0} &= \operatorname*{argmax}_s \max_{z_{1:T_0-1}} P(Z_{T_0} = s, Z_{1:T_0-1} = z_{1:T_0-1}, X_{1:T} = x_{1:T}) \\
&= \operatorname*{argmax}_s \max_{z_{1:T_0-1}} P(Z_{T_0} = s, Z_{1:T_0-1} = z_{1:T_0-1}, X_{1:T_0} = x_{1:T_0}) \times \\
&\qquad P(X_{T_0+1,T} = x_{T_0+1:T} \mid Z_{T_0} = s, Z_{1:T_0-1} = z_{1:T_0-1}, X_{1:T_0} = x_{1:T_0})
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&= \operatorname*{argmax}_s \left(\max_{z_{1:T_0-1}} P(Z_{T_0} = s, Z_{1:T_0-1} = z_{1:T_0-1}, X_{1:T_0} = x_{1:T_0}) \right) \times \\
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&= \operatorname*{argmax}_s \left(\max_{z_{1:T_0-1}} P(Z_{T_0} = s, Z_{1:T_0-1} = z_{1:T_0-1}, X_{1:T_0} = x_{1:T_0}) \right) \times \\
&\qquad P(X_{T_0+1,T} = x_{T_0+1:T} \mid Z_{T_0} = s) \\
&= \operatorname*{argmax}_s \ \delta_s(T_0)\beta_s(T_0)
\end{aligned}
$$

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Again, neither is true.

Exercise 3 (cont.)

Viterbi Algorithm with partial data $x_{1:T_0}$

For each $s \in [S]$, compute $\delta_s(1) = \pi_s b_{s,x_1}$.

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For each $t = 2, \ldots, T$,

- for each $s \in [S]$, compute

$$\delta_s(t) = \begin{cases} b_{s,x_t} \max_{s'} a_{s',s}\delta_{s'}(t-1) & \text{if } t \leq T_0 \\ & \end{cases}$$

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$$\Delta_s(t) = \operatorname*{argmax}_{s'} a_{s',s}\delta_{s'}(t-1).$$

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$$\Delta_s(t) = \operatorname*{argmax}_{s'} a_{s',s} \delta_{s'}(t-1).$$

Backtracking: let $z_T^* = \operatorname{argmax}_s \delta_s(T)$.

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$$\Delta_s(t) = \underset{s'}{\operatorname{argmax}}\, a_{s',s} \delta_{s'}(t-1).$$

Backtracking: let $z_T^* = \operatorname{argmax}_s \delta_s(T)$.
For each $t = T, \ldots, 2$: set $z_{t-1}^* = \Delta_{z_t^*}(t)$.

Output the most likely path $z_1^*, \ldots, z_T^*$.

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Difference compared to "most likely path $z^*_{1:T}$ given $x_{1:T}$":

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- most likely path $z^*_{1:T_0}$ given $x_{1:T_0}$
  - exact same δ, Δ
  - find the last state using $\delta_s(T_0)$

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The rest of the backtracking is always the same: **follow the arrows!**