CSCI567 Machine Learning (Fall 2025)

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University of Southern California

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Administration

Reminder: HW3 is due on Nov 5th.

Outline

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1. Principal Component Analysis (PCA)
2. Markov models
3. Hidden Markov Model

Dimensionality reduction

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There are many approaches, we focus on a linear method:
Principal Component Analysis (PCA)

Example

picture from here

Consider the following dataset:

- 17 features, each represents the average consumption of some food

Alcoholic drinks	375	135	458	475
Beverages	57	47	53	73
Carcase meat	245	267	242	227
Cereals	1472	1494	1462	1582
Cheese	105	66	103	103
Confectionery	54	41	62	64
Fats and oils	193	209	184	235
Fish	147	93	122	160
Fresh fruit	1102	674	957	1137
Fresh potatoes	720	1033	566	874
Fresh Veg	253	143	171	265
Other meat	685	586	750	803
Other Veg	488	355	418	570
Processed potatoes	198	187	220	203
Processed Veg	360	334	337	365
Soft drinks	1374	1506	1572	1256
Sugars	156	139	147	175

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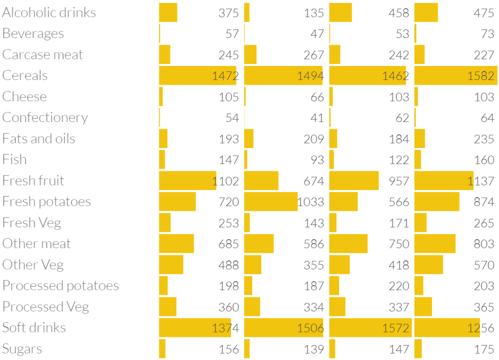
- 4 data points, each represents some country



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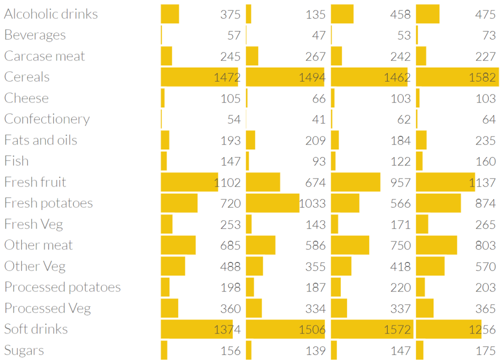

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Hard to say anything looking at all these 17 features.

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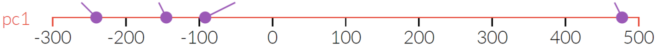

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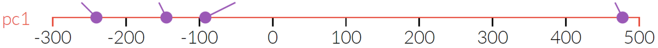

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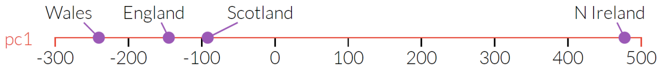

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Now one data point is clearly different from the rest!

That turns out to be data from Northern Ireland, *the only country not on the island of Great Britain out of the 4 samples.*

Example

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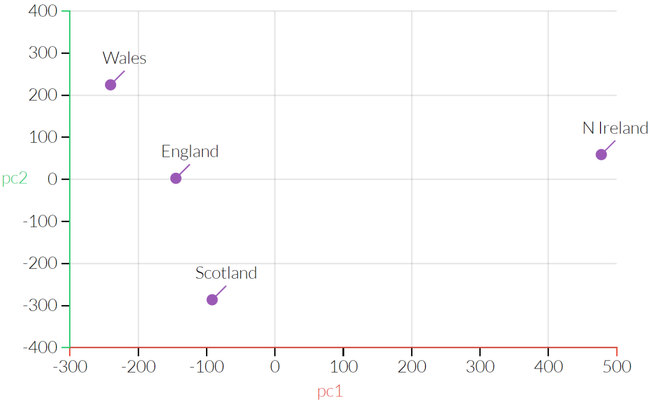
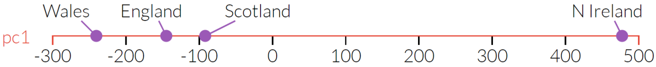
PCA can find the **second (and more) principal component** of the data too:



High level idea

How does PCA find these principal components (PC)?



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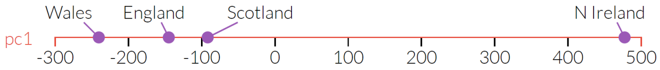

The first PC is in fact **the direction with the most variance**, i.e. the direction where the data is most spread out.

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- if we pre-center the data, i.e. let $\boldsymbol{x}'_n = \boldsymbol{x}_n - \frac{1}{N} \sum_m \boldsymbol{x}_m$, then the objective simply becomes

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- we will simply assume $\{\boldsymbol{x}_n\}$ is centered (to avoid notation $\boldsymbol{x}_n'$)

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With $\boldsymbol{X} \in \mathbb{R}^{N\times \mathsf{D}}$ being the data matrix (as in Lec 2), we want

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where the columns of U^{T} are unit **eigenvectors** $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{\mathsf{D}}$ of $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}$ with corresponding **eigenvalues** $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_{\mathsf{D}} \geq 0$.

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Conclusion: <u>the first PC is the top eigenvector of the covariance matrix!</u>

Finding the other PCs

If v_1 is the first PC, then the **second PC** is found via

$$\max_{\boldsymbol{v}_2 : \|\boldsymbol{v}_2\|_2 = 1, \boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{v}_2 = 0} \boldsymbol{v}_2^{\mathrm{T}} \left(\boldsymbol{X}^{\mathrm{T}} \boldsymbol{X} \right) \boldsymbol{v}_2$$

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Conclusion: the d-th principal component is the d-th eigenvector (sorted by the eigenvalue from largest to smallest).

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(Optional) Step 4 "Reconstruct" original dataset as $\boldsymbol{X}\boldsymbol{V}\boldsymbol{V}^{\mathrm{T}} \in \mathbb{R}^{N\times \mathsf{D}}$

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For visualization, also often pick $p = 1$ or $p = 2$.

Another visualization example

A famous study of **genetic map**

- dataset: genomes of 1,387 Europeans

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A famous study of **genetic map**

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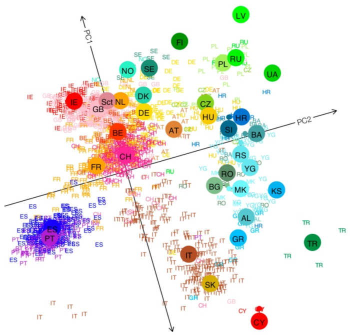
- first 2 PCs shown below;



Another visualization example

A famous study of **genetic map**

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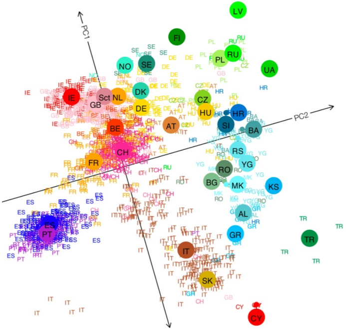

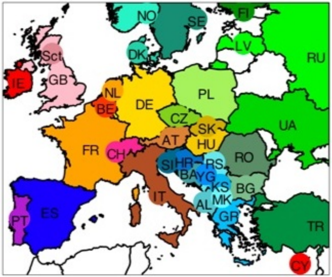

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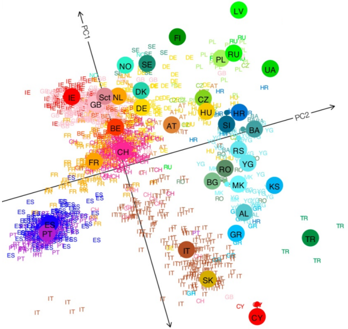

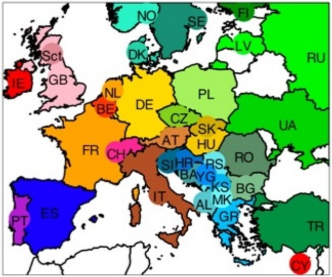

Genetic variation is highly structured by geography; top PCs correspond to geographic factors.

Compression Example

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PCA finds the subspace (out of the space of all 256×256 images) where faces lie

Figure 2. Seven of the eigenfaces calculated from the input images of Figure 1.

Word embedding via PCA

Create **meaningful vector representation** of words; see project.

Outline

Sequential data and Markov models

Sequential data (i.e., each $\boldsymbol{x}_n$ is a *sequence*) are ubiquitous:

- text or speech data
- stock market data
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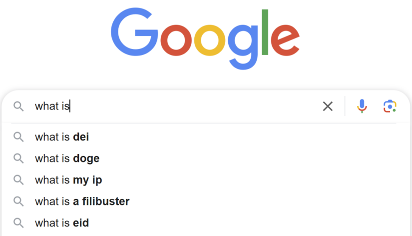
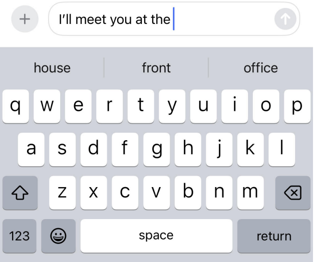
- but still extremely useful in many areas (in particular, it is the foundation of *reinforcement learning*)

A motivating example

Next word prediction





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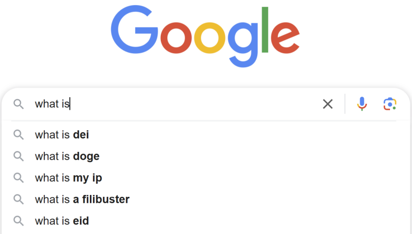

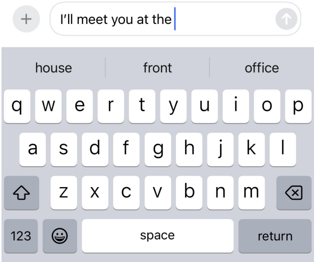

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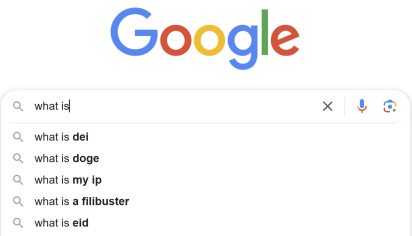

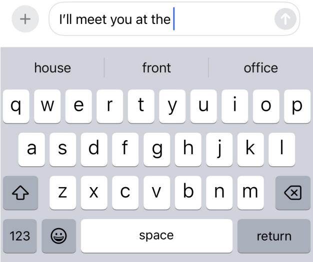

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abbreviated as $P(Z_{t+1} \mid Z_{1:t})$ (i.e., $Z_{1:t}$ denotes the sequence $Z_1, \ldots, Z_t$).

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- $(\{\pi_s\}, \{a_{s,s'}\}) = (\boldsymbol{\pi}, \boldsymbol{A})$ are parameters of the model

Examples

- Example 1 (**Language model**)

  States $[S]$ represent a dictionary of words,

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- Example 2 (**Weather**)

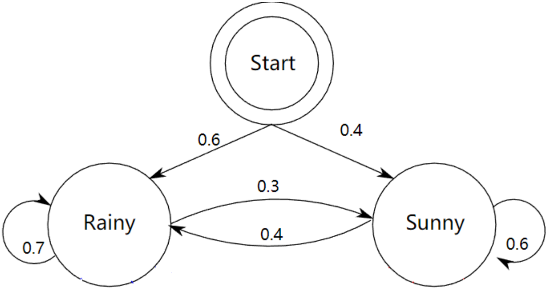
  States $[S]$ represent weather at each day

$$a_{\text{sunny,rainy}} = P(Z_{t+1} = \text{rainy} \mid Z_t = \text{sunny})$$

Graph representation

picture from Wikipedia

It is intuitive to represent a Markov model as a **graph**



High-order Markov chain

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Is the Markov assumption reasonable? Not completely for the language model for example:

$$P(Z_{t+1} = \text{office} \mid Z_{1:t} = \text{I'll meet you at the})$$
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Higher-order Markov chains make it more reasonable, e.g.

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i.e. the current word only depends on the last two words (the **trigram** model).

n-gram model

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For simplicity, we will only consider standard Markov chains.

Learning from examples

Now suppose we have observed N **sequences of examples**:

- $z_{1,1}, \ldots, z_{1,T}$
- $\cdots$
- $z_{n,1}, \ldots, z_{n,T}$
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From these observations how do we *learn the model parameters* $(\boldsymbol{\pi}, \boldsymbol{A})$?

Finding the MLE

Same story, find the **MLE**.

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Finding the MLE

Summing over all N sequences, we obtain the joint log-likelihood:

$$L(\boldsymbol{\pi}, \boldsymbol{A}) = \sum_{s} (\textbf{\#initial states with value } s) \ln \pi_s + \sum_{s,s'} (\textbf{\#transitions from } s \textbf{ to } s') \ln a_{s,s'}$$

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The MLE $\operatorname{argmax}_{\boldsymbol{\pi},\boldsymbol{A}} L(\boldsymbol{\pi}, \boldsymbol{A})$ is:

$$\pi_s \propto \#\text{initial states with value } s$$
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Example

Suppose we observed the following 2 sequences of length 5

- sunny, sunny, rainy, rainy, rainy
- rainy, sunny, sunny, sunny, rainy

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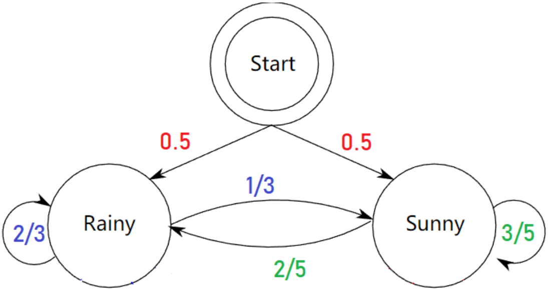
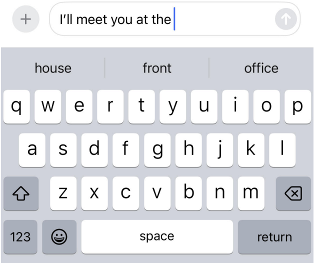
MLE is the following model



n-gram example

Recall: I'll meet you at the ______.



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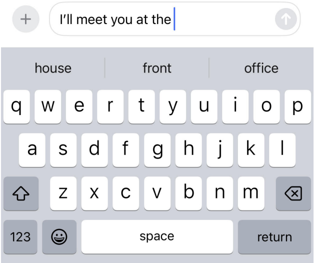

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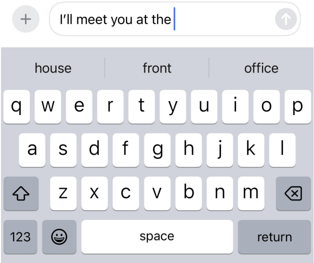

Recall: I'll meet you at the _____.

Suppose

- we use a 5-gram model
- "meet you at the" appears 1000 times in the training set:
  - "meet you at the house" appears 500 times
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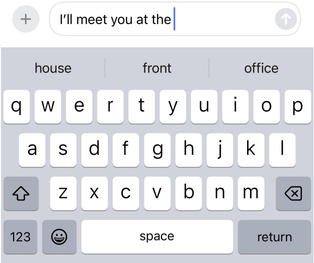

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Then the model predicts the next word as "house" with prob. 0.5, "front" with prob. 0.3, and "office" with prob. 0.2.

Text generation

adapted from Stanford CS224n

After learning the model, can also **generate texts** by repeatedly sampling conditioning on what have been generated:

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bank	0.153
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…	

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Text generation

adapted from Stanford CS224n

After learning the model, can also **generate texts** by repeatedly sampling conditioning on what have been generated:

- today the ______
- today the price ______
- today the price of ______
- today the price of gold ______

company	0.153
bank	0.153
$\boxed{\text{price}}$	0.077
italian	0.039
emirate	0.039
…	

$\boxed{\text{of}}$	0.308
for	0.050
it	0.046
to	0.046
is	0.031
…	

the	0.072
18	0.043
oil	0.043
its	0.036
$\boxed{\text{gold}}$	0.018
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final result: *today the price of gold per ton, while production of shoe lasts and shoe industry, the bank intervened just after it considered and rejected an IMF demand to rebuild depleted European stocks.*

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Surprisingly grammatical! but *incoherent*...

Outline

Markov model with outcomes/observations

Now suppose each state Z_t also "emits" some **outcome/observation** $X_t \in [O]$ based on the following model

$$P(X_t = o \mid Z_t = s) = b_{s,o} \qquad \text{(emission probability)}$$

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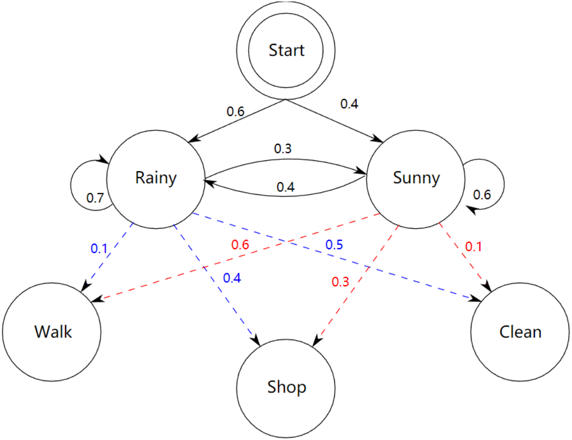
The model parameters are $(\{\pi_s\}, \{a_{s,s'}\}, \{b_{s,o}\}) = (\boldsymbol{\pi}, \boldsymbol{A}, \boldsymbol{B})$.

Another (toy) running example

picture from Wikipedia

On each day, we also observe **Bob's activity: walk, shop, or clean**, which only depends on the weather of that day.



Joint likelihood

The joint log-likelihood of a state-outcome sequence $z_1, x_1, \ldots, z_T, x_T$ is

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&\qquad \text{(due to all the independence)} \\
&= \ln \pi_{z_1} + \sum_{t=2}^{T} \ln a_{z_{t-1}, z_t} + \sum_{t=1}^{T} \ln b_{z_t, x_t}
\end{aligned}$$

Learning the model

If we observe N state-outcome sequences: $z_{n,1}, x_{n,1}, \ldots, z_{n,T}, x_{n,T}$ for $n = 1, \ldots, N$, the MLE is again very simple (verify yourself):

$$\pi_s \propto \textbf{\#initial states with value } s$$
$$a_{s,s'} \propto \textbf{\#transitions from } s \textbf{ to } s'$$
$$b_{s,o} \propto \textbf{\#state-outcome pairs } (s, o)$$

Hidden Markov models

However, *most often we do not observe the states!* Hence the name **Hidden Markov Model (HMM)**

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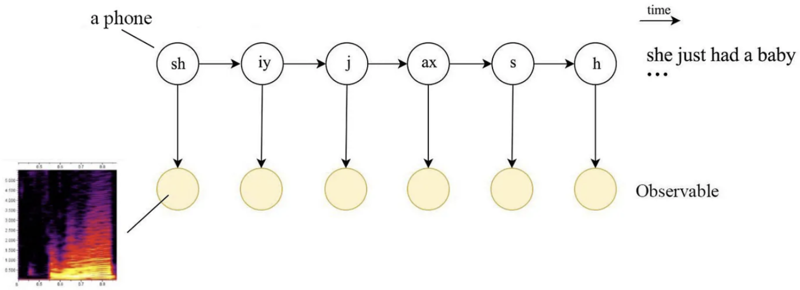
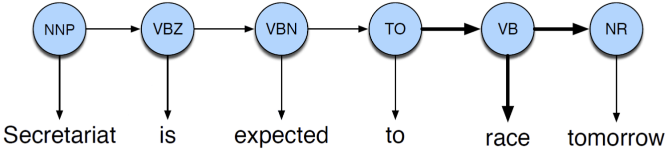
- **speech recognition**: observe the speech $X_{1:T}$ but not the underlying words/phones $Z_{1:T}$



Hidden Markov models (cont.)

- **Part of Speech (POS) tagging**: observe the sentence $X_{1:T}$ but not the underlying categories $Z_{1:T}$ (noun, verb, adjective, adverb, etc.)



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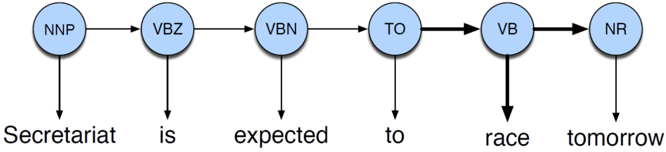

See programming project!

Learning HMMs

How to learn HMMs?

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How to learn HMMs? **Roadmap**:

- first discuss how to **infer** when the model is known (key: dynamic programming)

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- first discuss how to **infer** when the model is known (key: dynamic programming)
- then discuss how to **learn** the model (key: EM)

What can we infer about an HMM?

Knowing the parameter of an HMM, we can infer

- **most likely hidden states path, given an observation sequence**

$$\operatorname*{argmax}_{z_{1:T}} P(Z_{1:T} = z_{1:T} \mid X_{1:T} = x_{1:T})$$

  e.g. given Bob's activities for one week, what's the most likely weather for this week?

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- **the probability of observing some sequence**

$$P(X_{1:T} = x_{1:T})$$

  e.g. prob. of observing Bob's activities "walk, walk, shop, clean, walk, shop, shop" for one week

What can we infer for a known HMM?

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- **the state at some point, given an observation sequence**

$$P(Z_t = s \mid X_{1:T} = x_{1:T})$$

  e.g. given Bob's activities for one week, how was the weather like on Wed?

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  e.g. given Bob's activities for one week, how was the weather like on Wed?

- **the transition at some point, given an observation sequence**

$$P(Z_t = s, Z_{t+1} = s' \mid X_{1:T} = x_{1:T})$$

  e.g. given Bob's activities for one week, how was the weather like on Wed and Thu?

Forward and backward messages

The key to infer all these is to compute two things:

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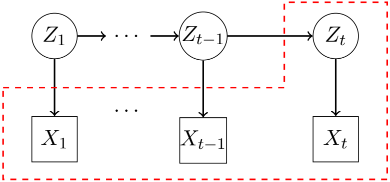
- **backward messages**: for each s and t

$$\beta_s(t) = P(X_{t+1:T} = x_{t+1:T} \mid Z_t = s)$$

Computing forward messages

Key: *establish recursion*

$$\alpha_s(t) = P(Z_t = s, X_{1:t} = x_{1:t})$$



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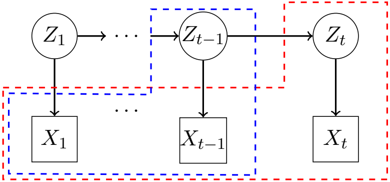

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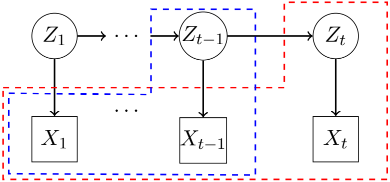

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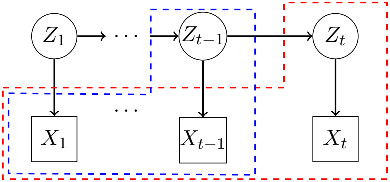

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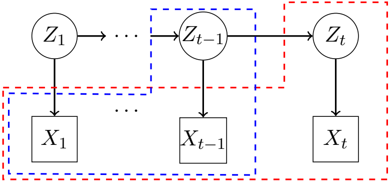

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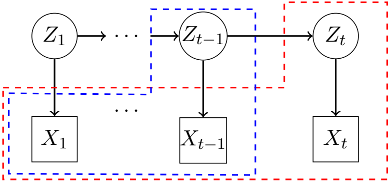

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Base case: $\alpha_s(1) = P(Z_1 = s, X_1 = x_1) = \pi_s b_{s,x_1}$

Forward procedure

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For all $s \in [S]$, compute $\alpha_s(1) = \pi_s b_{s,x_1}$.

For $t = 2, \ldots, T$

- for each $s \in [S]$, compute

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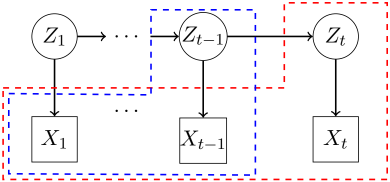

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It takes

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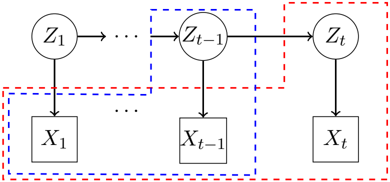
- $O(ST)$ space



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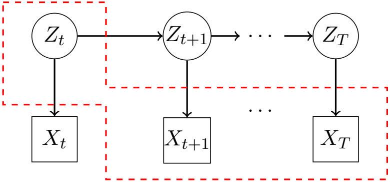

Again, establish a recursion

$$\beta_s(t)$$
$$= P(X_{t+1:T} = x_{t+1:T} \mid Z_t = s)$$

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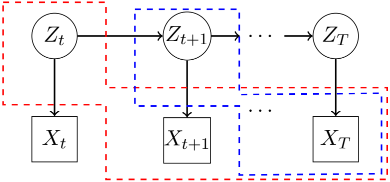

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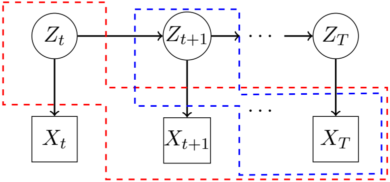

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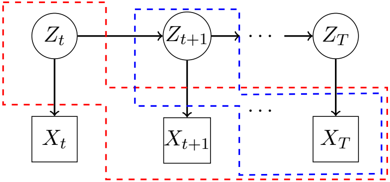

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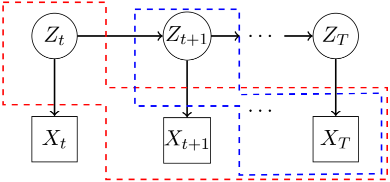

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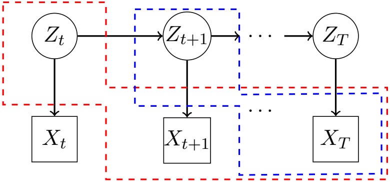

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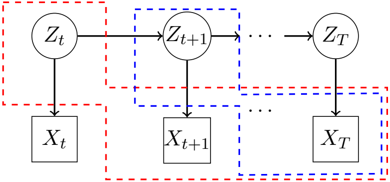

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Base case: $\beta_s(T) = 1$

Backward procedure

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For $t = T-1, \ldots, 1$

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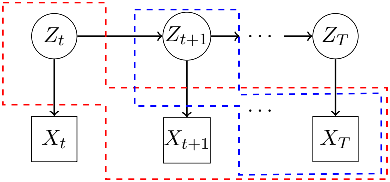

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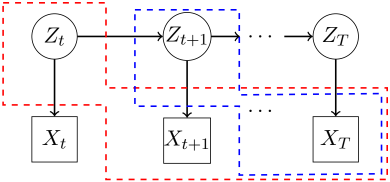

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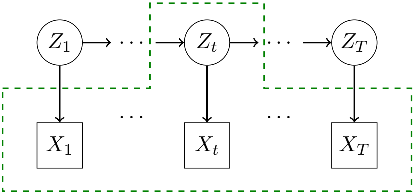

$$\gamma_s(t) = P(Z_t = s \mid X_{1:T} = x_{1:T})$$

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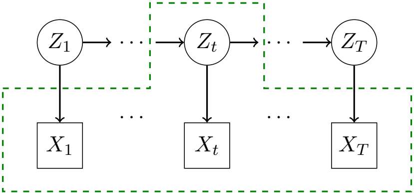

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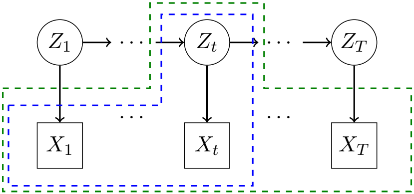

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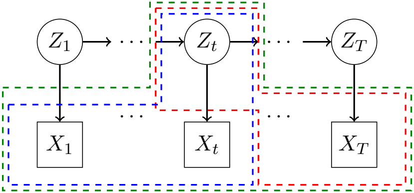
\end{aligned}$$

Using forward and backward messages

With forward and backward messages, we can easily infer many things, e.g.



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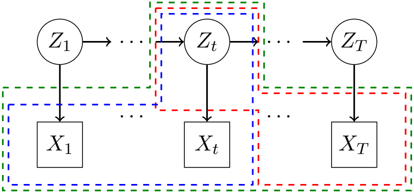

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What constant are we omitting in "$\propto$"?

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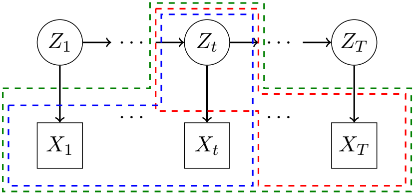

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$$P(X_{1:T} = x_{1:T}) = \sum_s \alpha_s(t)\beta_s(t),$$

the probability of observing the sequence $x_{1:T}$.

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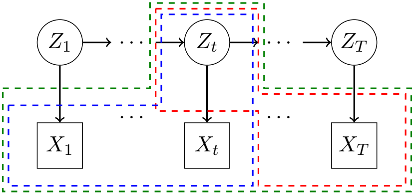

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$$P(X_{1:T} = x_{1:T}) = \sum_s \alpha_s(t)\beta_s(t),$$

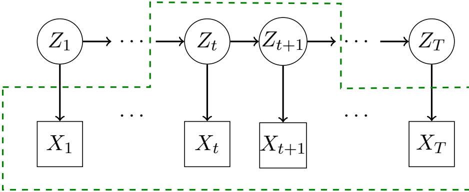
the probability of observing the sequence $x_{1:T}$.

This is true for any t; a good way to check correctness of your code.

Using forward and backward messages

Another example: the conditional probability of transition s to s' at time t

$$\xi_{s,s'}(t) = P(Z_t = s, Z_{t+1} = s' \mid X_{1:T} = x_{1:T})$$



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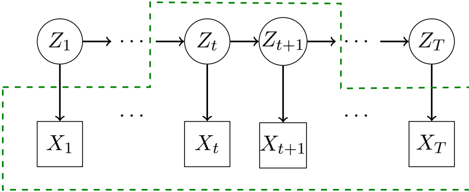

$$\xi_{s,s'}(t)$$
$$= P(Z_t = s, Z_{t+1} = s' \mid X_{1:T} = x_{1:T})$$
$$\propto P(Z_t = s, Z_{t+1} = s', X_{1:T} = x_{1:T})$$

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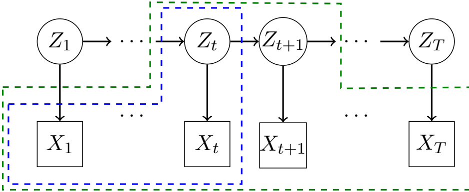

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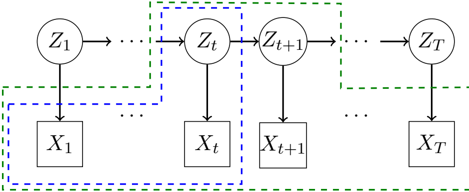

$$
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\end{aligned}
$$

Using forward and backward messages

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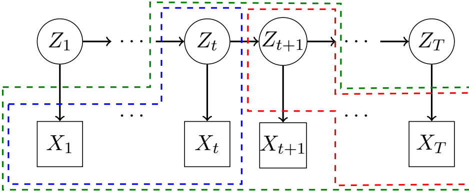

$$
\begin{aligned}
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$$

Using forward and backward messages

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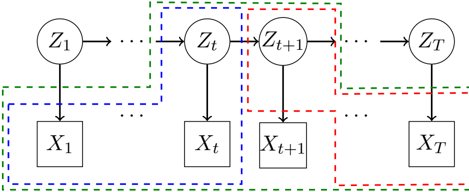

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&= \alpha_s(t) a_{s,s'} b_{s',x_{t+1}} \beta_{s'}(t+1)
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Using forward and backward messages

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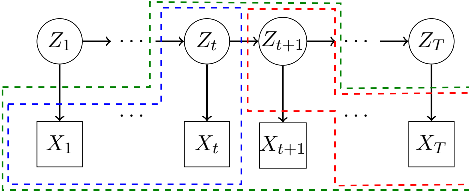

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\end{aligned}
$$

The normalization constant is in fact again $P(X_{1:T} = x_{1:T})$

Finding the most likely path

Can't use forward and backward messages directly to find the most likely path, but it is very similar to the forward procedure.

Finding the most likely path

Can't use forward and backward messages directly to find the most likely path, but it is very similar to the forward procedure. Key: compute

$$\delta_s(t) = \max_{z_{1:t-1}} P(Z_t = s, Z_{1:t-1} = z_{1:t-1}, X_{1:t} = x_{1:t})$$

the probability of the most likely path for time $1:t$ ending at state s

Computing $\delta_s(t)$

Observe

$$\delta_s(t) = \max_{z_{1:t-1}} P(Z_t = s, Z_{1:t-1} = z_{1:t-1}, X_{1:t} = x_{1:t})$$

Computing $\delta_s(t)$

Observe

$$\begin{aligned}\delta_s(t) &= \max_{z_{1:t-1}} P(Z_t = s, Z_{1:t-1} = z_{1:t-1}, X_{1:t} = x_{1:t}) \\ &= \max_{s'} \max_{z_{1:t-2}} P(Z_t = s, Z_{t-1} = s', Z_{1:t-2} = z_{1:t-2}, X_{1:t} = x_{1:t})\end{aligned}$$

Computing $\delta_s(t)$

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&= b_{s,x_t} \max_{s'} a_{s',s} \delta_{s'}(t-1) \qquad \text{(recursion!)}
\end{aligned}
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Computing $\delta_s(t)$

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\end{aligned}$$

Base case: $\delta_s(1) = P(Z_1 = s, X_1 = x_1) = \pi_s b_{s,x_1}$

Computing $\delta_s(t)$

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\end{aligned}
$$

Base case: $\delta_s(1) = P(Z_1 = s, X_1 = x_1) = \pi_s b_{s,x_1}$

Exactly the same as forward messages except replacing "sum" by "max"!

Viterbi Algorithm (!)

Viterbi Algorithm

For each $s \in [S]$, compute $\delta_s(1) = \pi_s b_{s,x_1}$.

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For each $s \in [S]$, compute $\delta_s(1) = \pi_s b_{s,x_1}$.

For each $t = 2, \ldots, T$,

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$$\delta_s(t) = b_{s,x_t} \max_{s'} a_{s',s} \delta_{s'}(t-1),$$

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Backtracking: let $z_T^* = \operatorname{argmax}_s \delta_s(T)$.

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Backtracking: let $z_T^* = \operatorname{argmax}_s \delta_s(T)$.
For each $t = T, \ldots, 2$: set $z_{t-1}^* = \Delta_{z_t^*}(t)$.

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Backtracking: let $z_T^* = \mathrm{argmax}_s\, \delta_s(T)$.
For each $t = T, \ldots, 2$: set $z_{t-1}^* = \Delta_{z_t^*}(t)$.

Output the most likely path $z_1^*, \ldots, z_T^*$.

Example: $x_{1:4}$ = clean, shop, walk, walk

Viterbi Algorithm

For each $s \in [S]$, compute $\delta_s(1) = \pi_s b_{s,x_1}$.

...

Example: $x_{1:4} =$ clean, shop, walk, walk

Viterbi Algorithm

For each $s \in [S]$, compute $\delta_s(1) = \pi_s b_{s,x_1}$.

. . .

$$\delta_{\text{sunny}}(1) = \pi_{\text{sunny}} b_{\text{sunny},\text{clean}}$$

Example: $x_{1:4} =$ clean, shop, walk, walk

Viterbi Algorithm

For each $s \in [S]$, compute $\delta_s(1) = \pi_s b_{s,x_1}$.

$\ldots$

$$\delta_{\text{sunny}}(1) = \pi_{\text{sunny}} b_{\text{sunny},\text{clean}}$$

$$\delta_{\text{rainy}}(1) = \pi_{\text{rainy}} b_{\text{rainy},\text{clean}}$$

Example: $x_{1:4}$ = clean, shop, walk, walk

Viterbi Algorithm

For each $s \in [S]$, compute $\delta_s(1) = \pi_s b_{s,x_1}$.

...

$$\delta_{\text{sunny}}(1) = \pi_{\text{sunny}} b_{\text{sunny},\text{clean}}$$

$$\delta_{\text{rainy}}(1) = \pi_{\text{rainy}} b_{\text{rainy},\text{clean}}$$

$$\boxed{\delta_{\text{sunny}}(1) = 0.25}$$

$$\boxed{\delta_{\text{rainy}}(1) = 0.4}$$

(numbers are made up here)

Example: $x_{1:4}$ = clean, shop, walk, walk

Viterbi Algorithm

...

For each $t = 2, \ldots, T$ and each $s \in [S]$, compute

$$\delta_s(t) = b_{s,x_t} \max_{s'} a_{s',s}\delta_{s'}(t-1), \quad \Delta_s(t) = \operatorname*{argmax}_{s'} a_{s',s}\delta_{s'}(t-1).$$

$\boxed{\delta_{\text{sunny}}(1) = 0.25}$

$\boxed{\delta_{\text{rainy}}(1) = 0.4}$

Example: $x_{1:4} =$ clean, shop, walk, walk

Viterbi Algorithm

...

For each $t = 2, \ldots, T$ and each $s \in [S]$, compute

$$\delta_s(t) = b_{s,x_t} \max_{s'} a_{s',s}\delta_{s'}(t-1), \quad \Delta_s(t) = \operatorname*{argmax}_{s'} a_{s',s}\delta_{s'}(t-1).$$

$$\delta_{\text{sunny}}(2) = b_{\text{sunny,shop}} \max_{s' \in \{\text{sunny,rainy}\}} a_{s',\text{sunny}}\delta_{s'}(1)$$

$\boxed{\delta_{\text{sunny}}(1) = 0.25}$

$\boxed{\delta_{\text{rainy}}(1) = 0.4}$

Example: $x_{1:4}$ = clean, shop, walk, walk

Viterbi Algorithm

...

For each $t = 2, \ldots, T$ and each $s \in [S]$, compute

$$\delta_s(t) = b_{s,x_t} \max_{s'} a_{s',s}\delta_{s'}(t-1), \quad \Delta_s(t) = \operatorname*{argmax}_{s'} a_{s',s}\delta_{s'}(t-1).$$

$$\delta_{\text{sunny}}(2) = b_{\text{sunny},\text{shop}} \max_{s' \in \{\text{sunny},\text{rainy}\}} a_{s',\text{sunny}}\delta_{s'}(1)$$

$$\delta_{\text{rainy}}(2) = b_{\text{rainy},\text{shop}} \max_{s' \in \{\text{sunny},\text{rainy}\}} a_{s',\text{rainy}}\delta_{s'}(1)$$

$\boxed{\delta_{\text{sunny}}(1) = 0.25}$

$\boxed{\delta_{\text{rainy}}(1) = 0.4}$

Example: $x_{1:4} =$ clean, shop, walk, walk

Viterbi Algorithm

...

For each $t = 2, \ldots, T$ and each $s \in [S]$, compute

$$\delta_s(t) = b_{s,x_t} \max_{s'} a_{s',s}\delta_{s'}(t-1), \quad \Delta_s(t) = \operatorname*{argmax}_{s'} a_{s',s}\delta_{s'}(t-1).$$

$$\delta_{\text{sunny}}(2) = b_{\text{sunny,shop}} \max_{s' \in \{\text{sunny,rainy}\}} a_{s',\text{sunny}}\delta_{s'}(1)$$

$$\delta_{\text{rainy}}(2) = b_{\text{rainy,shop}} \max_{s' \in \{\text{sunny,rainy}\}} a_{s',\text{rainy}}\delta_{s'}(1)$$

$\boxed{\delta_{\text{sunny}}(1) = 0.25}$ $\boxed{\delta_{\text{sunny}}(2) = 0.1}$

$\boxed{\delta_{\text{rainy}}(1) = 0.4}$ $\boxed{\delta_{\text{rainy}}(2) = 0.19}$

Example: $x_{1:4}$ = clean, shop, walk, walk

Viterbi Algorithm

...

For each $t = 2, \ldots, T$ and each $s \in [S]$, compute

$$\delta_s(t) = b_{s,x_t} \max_{s'} a_{s',s}\delta_{s'}(t-1), \quad \Delta_s(t) = \operatorname*{argmax}_{s'} a_{s',s}\delta_{s'}(t-1).$$

$$\delta_{\text{sunny}}(2) = b_{\text{sunny,shop}} \max_{s' \in \{\text{sunny},\text{rainy}\}} a_{s',\text{sunny}}\delta_{s'}(1)$$

$$\delta_{\text{rainy}}(2) = b_{\text{rainy,shop}} \max_{s' \in \{\text{sunny},\text{rainy}\}} a_{s',\text{rainy}}\delta_{s'}(1)$$

$$\begin{tikzcd}
\boxed{\delta_{\text{sunny}}(1) = 0.25} & \boxed{\delta_{\text{sunny}}(2) = 0.1} \arrow[l] \\
\boxed{\delta_{\text{rainy}}(1) = 0.4} & \boxed{\delta_{\text{rainy}}(2) = 0.19} \arrow[l]
\end{tikzcd}$$

Arrows represent the "argmax", i.e. $\Delta_s(t)$.

Example: $x_{1:4}$ = clean, shop, walk, walk

Viterbi Algorithm

...

For each $t = 2, \ldots, T$ and each $s \in [S]$, compute

$$\delta_s(t) = b_{s,x_t} \max_{s'} a_{s',s}\delta_{s'}(t-1), \quad \Delta_s(t) = \operatorname*{argmax}_{s'} a_{s',s}\delta_{s'}(t-1).$$

$$\delta_{\text{sunny}}(3) = b_{\text{sunny,walk}} \max_{s' \in \{\text{sunny,rainy}\}} a_{s',\text{sunny}}\delta_{s'}(2)$$

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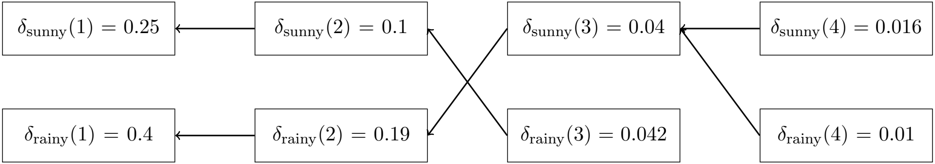

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Example: $x_{1:4}$ = clean, shop, walk, walk

Viterbi Algorithm

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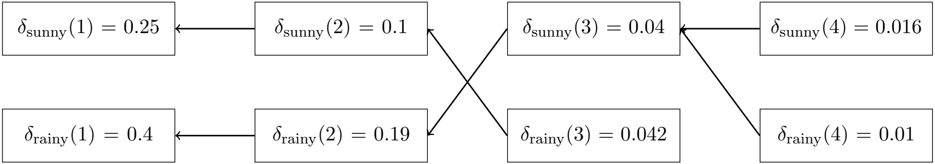
Backtracking: let $z_T^* = \operatorname{argmax}_s \delta_s(T)$.

For each $t = T, \ldots, 2$: set $z_{t-1}^* = \Delta_{z_t^*}(t)$.

Output the most likely path $z_1^*, \ldots, z_T^*$.



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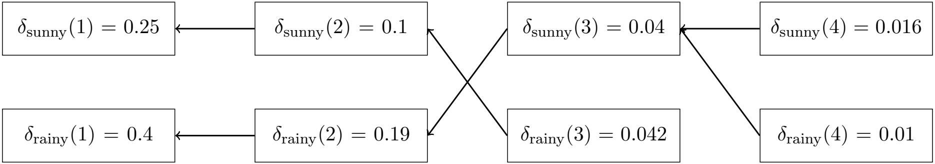

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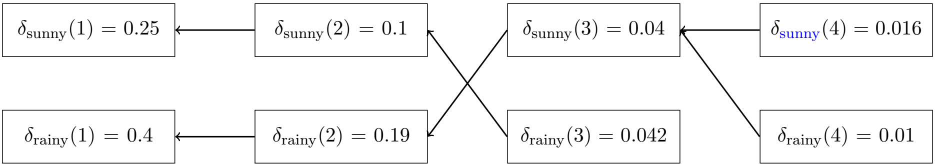
$$z_4^* = \underset{s \in \{\text{sunny}, \text{rainy}\}}{\operatorname{argmax}} \delta_s(4) = \text{sunny},$$



The most likely path is "**sunny**"

Example: $x_{1:4}$ = clean, shop, walk, walk

Viterbi Algorithm

$\ldots$

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Output the most likely path $z_1^*, \ldots, z_T^*$.

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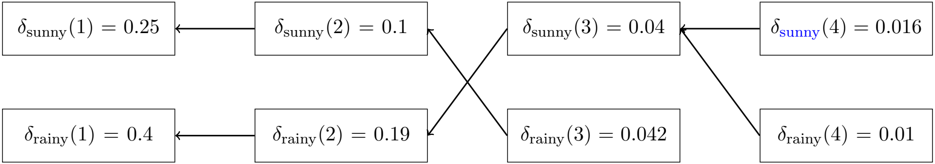

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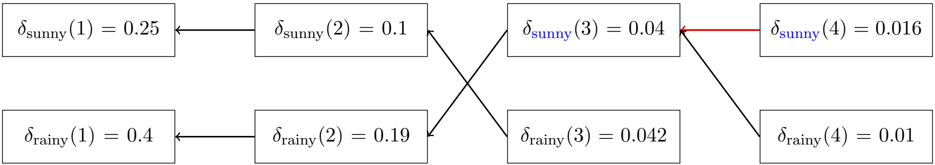

The most likely path is **“ sunny, sunny”**

Example: $x_{1:4}$ = clean, shop, walk, walk

Viterbi Algorithm

. . .

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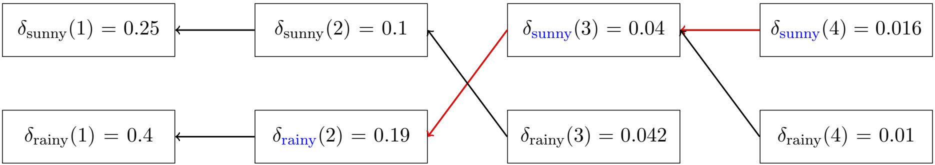

The most likely path is "**rainy, sunny, sunny**"

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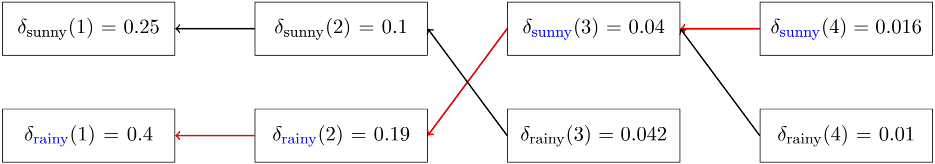

The most likely path is **"rainy, rainy, sunny, sunny"**

Learning the parameters of an HMM

All previous inferences depend on knowing the parameters $(\boldsymbol{\pi}, \boldsymbol{A}, \boldsymbol{B})$.

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Need to apply **EM** again! Known as the **Baum–Welch algorithm**.

Applying EM: E-Step

Recall in the E-Step we fix the parameters and find the **posterior distributions q of the hidden states** (for each sample n),

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$$
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&= \mathbb{E}_{z_{1:T}\sim q}\left[\ln \pi_{z_1} + \sum_{t=1}^{T-1} \ln a_{z_t,z_{t+1}} + \sum_{t=1}^{T} \ln b_{z_t,x_t}\right] \\
&= \sum_s \gamma_s(1)\ln \pi_s + \sum_{t=1}^{T-1}\sum_{s,s'} \xi_{s,s'}(t)\ln a_{s,s'} + \sum_{t=1}^{T}\sum_s \gamma_s(t)\ln b_{s,x_t}
\end{aligned}
$$

We have discussed how to compute

$$
\begin{aligned}
\gamma_s(t) &= P(Z_t = s \mid X_{1:T}=x_{1:T}) \\
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Applying EM: M-Step

The maximizer of complete log-likelihood is simply doing **weighted counting** (compared to the unweighted counting on Slide 36):

$$\pi_s \propto \sum_n \gamma_s^{(n)}(1) = \mathbb{E}_q\left[\ \textbf{\#initial states with value } s\right]$$

$$a_{s,s'} \propto \sum_n \sum_{t=1}^{T-1} \xi_{s,s'}^{(n)}(t) = \mathbb{E}_q\left[\ \textbf{\#transitions from } s \textbf{ to } s'\right]$$

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(Recall how EM for GMM updates parameters using weighted averages.)

Baum–Welch algorithm

Step 0 Initialize the parameters $(\boldsymbol{\pi}, \boldsymbol{A}, \boldsymbol{B})$

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Step 1 (E-Step) Fixing the parameters, compute forward and backward messages for all sample sequences, then use these to compute $\gamma_s^{(n)}(t)$ and $\xi_{s,s'}^{(n)}(t)$ for each n, t, s, s' (see Slides 47 and 48).

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Step 3 Return to Step 1 if not converged

Summary

Very important models: **Markov chains**, **hidden Markov models**

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Several algorithms:

- forward and backward procedures
- inferring HMMs based on forward and backward messages
- Viterbi algorithm and variants (see today's discussion)
- Baum–Welch algorithm